Fundamentals of Laser Interactions
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FOREWORD

The Seminar on Fundamentals of Laser Interactions was the second Meeting on Laser Phenomena held at the Bundessportheim in Obergurgl. It was attended by 46 Physicists from Austria, Belgium, The Federal Republic of Germany, Finland, France, Hungary, Italy, Japan, The Netherlands, and the United States, who work actively in the rapidly developing field of laser interactions.

The Seminar presented an opportunity to discuss at leisure problems of mutual interest to theoreticians and experimentalists who are working on various aspects of the field of laser interactions. There was an attempt to bring together people who are doing research on multiphoton physics, on scattering phenomena, on many body problems, and on new methods of experimentation. In particular the following topics were chosen for discussion:

1) Multiphoton Spectroscopy
2) Electron Correlations in Multiphoton Transitions
3) Multiphoton Continuum Effects
4) Rydberg States in Strong Laser Fields
5) Laser Induced and Laser Assisted Scattering and Reactions
6) High Frequency Lasers
7) Laser Cooling and Trapping of Particles
8) Other Fundamental Interaction Processes

At the Seminar 18 Invited Lectures were given by:

G. Alber (JILA) 
N. Andersen (Copenhagen) 
W.E. Cooke (USC) 
M. Crance (Orsay) 
F.H.M. Faisal (Bielefeld) 
G. Ferrante (Palermo) 
M. Gavrila (Amsterdam) 
J. Javanainen (Helsinki) 
C.J. Joachain (Bruxelles) 

H. Helm (Menlo Park) 
H. Klar (Freiburg) 
L.A. Lompré (Saclay) 
C.K. Rhodes (Chicago) 
F. Roussel (Saclay) 
P.E. Toschek (Hamburg) 
C.R. Vidal (MPI Garching) 
H. Walther (MPI Garching) 
K.H. Welge (Bielefeld)

In addition, there were 13 contributed papers presented at the meeting.
The following pages present the full text of the invited lectures and the abstracts of the contributed papers. The invited lecture of V.G. Minogin (Moscow) was not presented at the Seminar but has been accepted for publication in the Proceedings. The editor is grateful to the contributors for their collaboration in preparing their typescripts for rapid publication.

The active yet relaxed atmosphere of the Bundessportheim at Obergurgl, surrounded by the snow-capped peaks of the Ötztal Alps, supplied a congenial setting for a very stimulating and rewarding meeting. It is a pleasure to thank all participants for their interest and enthusiasm. The most valuable secretarial assistance of Miss G. Eder is gratefully acknowledged.

Innsbruck, April 1985

F. Ehlotzky
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INVITED LECTURES

PART I: Collisions in Laser Fields
ELECTRON-ATOM INTERACTIONS IN INTENSE, HIGH FREQUENCY LASER FIELDS

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I. INTRODUCTION

Substantial effort has been invested in the development of very intense lasers, yielding about $10^{16}$ W/cm$^2$, and operated in a range of frequencies extending from the IR to the VUV. At these high intensities atomic transitions abundantly involve multiphoton absorption and emission (for a review of these processes see ref. 1). The description by perturbation theory is no longer valid, and new methods of solution of the Schrödinger equation are needed. A nonperturbative theory was developed earlier by Kroll and Watson for the low-frequency regime$^{2,1}$, well suited for the range of the intense IR lasers. We have recently developed a nonperturbative approach to deal with the opposite case, of the high-frequency regime$^{3,4,1}$. It applies to the intense excimer lasers already in operation in the VUV (e.g. see refs. 5,6), but extends beyond, into the XUV range.

In the following we shall present our theory for the high-frequency regime. We shall mainly deal with the case of electron-atom (ion) collisions in the radiation field, also termed free-free transitions. We will first describe the formalism (Sec.II), and then apply it to the case of a purely Coulomb potential (Sec.III). Further, in Sec.IV we will outline the extension of the method to encompass atomic structure and multiphoton ionization. Finally, in Sec.V we draw some conclusions.

II. FREE-FREE TRANSITIONS FORMALISM

A fully realistic description of the target atom is quite difficult. We shall represent it here by a potential model. (Very recently, however, we have extended the theory to take into account also the internal degrees of freedom). The potential will be taken to be of the central self-consistent type: Coulomb-like at the origin ($V(r)\sim-Z/r$), short range or ionic ($V(r)\sim-Z^+/r$) at large distances, but unspecified otherwise.

The laser field will be represented by a monochromatic infinite plane wave, linearly polarized, in the dipole approximation. The plane-wave assumption is not critical, as the extension to a single-mode laser pulse of adiabatically varying intensity can subsequently be made$^7$. Linear polarization is assumed in view of simplifying the algebra, and the dipole approximation is justified in the frequency range we are interested in (from the visible to the extreme ultraviolet). Con-
sequently, we take the electrodynamic potentials of the wave in the form
\[ \vec{A} = \vec{a} \cos \omega t \] (with \( \vec{a} \) real) and \( \phi = 0 \). (Note that our premises are the same as those of Kroll and Watson.\(^2\))

Application of the space translation transformation (Kramers\(^8\), Henneberger\(^9\)) to the Schrödinger equation gives
\[
[\hat{\mathbf{p}}^2 + V(\vec{r} + \vec{a}(t))]\psi = i(\partial \psi / \partial t),
\]
where
\[
\vec{a}(t) = -c^{-1} \int_0^t \vec{\dot{A}}(t') \, dt' = \vec{a}_0 \sin \omega t,
\]
\[
\vec{a}_0 = -\vec{a}_0 \hat{\mathbf{e}}, \quad \alpha_0 = a/\omega c = i^\dagger \omega^{-2},
\]
and \( \hat{\mathbf{e}} \) and \( I \) are the real polarization vector and (time averaged) intensity of the plane wave. All our formulas are written in atomic units; the a.u. of (time averaged) intensity is \( I_0 = 3.51 \times 10^{16} \text{ W/cm}^2 \).

Eq.(1) should be solved by imposing the boundary conditions of our problem: an incoming current of particles of energy \( E = p^2/2 \), and radially outgoing currents of scattered particles of energies and momenta
\[
E_n = E + n\omega, \quad \vec{p}_n = \vec{p}^2_n/2, \quad n = 0, \pm 1, \pm 2, \ldots.
\]

Equation (1) has periodic time-dependent coefficients. As usual, we seek a quasiperiodic solution of the form
\[
\psi(\vec{r},t) = e^{-iEt} \sum_{n=-\infty}^{+\infty} \psi_n(\vec{r}) e^{-in\omega t}.
\]
Then, we Fourier analyze the potential:
\[
V(\vec{r} + \vec{a}(t)) = \sum_{n=-\infty}^{+\infty} V_n(\vec{a}_0; \vec{r}) e^{-in\omega t}.
\]
By some algebraic manipulations the coefficients can be written as
\[
V_n(\vec{a}_0; \vec{r}) = (i\pi/n) \int_{-1}^{+1} V(\vec{r} + \vec{a}_0 u) T_n(u)(1-u^2)^{-1/2} \, du,
\]
where \( T_n(u) \) are Chebyshev polynomials.

Insertion of Eqs.(4) and (5) into Eq.(1) leads to a system of coupled differential equations for the components \( \psi_n(\vec{r}) \), which we write
\[
[i\hat{\mathbf{p}}^2 + V_0 - (E+n\omega)]\psi_n = -\sum_{m=-\infty}^{+\infty} V_{n-m}\psi_m.
\]

The boundary conditions require that our solutions \( \psi_n(\alpha_0, \omega; \vec{r}) \) behave asymptotically as follows:
\[ \psi_o(\vec{\alpha}_o, \omega; \vec{r}) = \exp \left\{ i(\vec{p} \cdot \vec{r} + \gamma_o \ln(\vec{p} \cdot \vec{r})) + \frac{1}{\vec{r}} f_o(\vec{\alpha}_o, \omega; \vec{r}) \exp[i(\vec{p} \cdot \vec{r} - \gamma_o \ln 2\vec{p} \cdot \vec{r})] \right\}, \quad (8) \]

\[ \psi_n(\vec{\alpha}_o, \omega; \vec{r}) = \frac{1}{\vec{r}} f_n(\vec{\alpha}_o, \omega; \vec{r}) \exp[i(p_n r - \gamma_n \ln 2p_n r)] \quad (n \neq 0), \quad (9) \]

with \( \gamma_n = -Z'/p_n \) (for a short-range potential \( Z' = 0 \)). Equation (8) contains the elastic scattering amplitude \( f_o(\vec{\alpha}_o, \omega; \vec{r}) \), and Eq. (9) that for absorption/emission \( f_n(\vec{\alpha}_o, \omega; \vec{r}) \). The associated scattering cross sections are

\[ d\sigma_n/d\Omega = \left( \frac{p_n}{p} \right) |f_n(\vec{\alpha}_o, \omega; \vec{r})|^2 \quad (n = 0, \pm 1, \pm 2, \ldots). \quad (10) \]

For a single-mode laser pulse of adiabatically varying intensity, Eq. (10) should be time-averaged appropriately. (7).

We shall now describe a method for handling the system Eq. (7). The left-hand side contains the Hamiltonian

\[ H = \frac{1}{2} \vec{p}^2 + V_o(\vec{\alpha}_o, \vec{r}). \quad (11) \]

By use of the Green's operator \( G(n) \) associated to it, where \( \Omega \) is the energy parameter, Eq. (7) may be formally solved as

\[ \psi_n = \psi_n^{(+)} \delta_{n_0} - G(+) (E_n) \sum m \frac{V_{n-m} \psi_m}{(m \neq n)} \quad (12) \]

Here \( \psi_n^{(+)} \) is the (\( \vec{\alpha}_o \)-dependent) solution of the equation

\[ H \psi_n^{(+)} = E \psi_n^{(+)} \quad (13) \]

satisfying the boundary condition Eq. (8) with an amplitude \( f_o(\vec{\alpha}_o; \vec{r}) \). It then follows from Eq. (12) that the \( \psi_n \) satisfy the boundary condition required by Eqs. (8) and (9) with the following expression for the scattering amplitudes:

\[ f_n(\vec{\alpha}_o, \omega; \vec{r}) = f_o(\vec{\alpha}_o; \vec{r}) \delta_{n_0} - (1 - \delta_{n_0}) \frac{1}{Z_{\pi}} \psi_n^{(-)} |V_n| \psi_n^{(+)} + \]

\[ + \frac{1}{Z_{\pi}} \sum_{m} \sum_{m'} \frac{V_{n-m} G^+(E_m) V_{m-m'} |\psi_{m'}|}{(m \neq n)(m' \neq m)} \quad (14) \]

Besides \( \psi_n^{(+)} \), Eq. (14) also contains \( \psi_n^{(-)} \), which is an incoming-wave solution of Eq. (13), as well as the unknown set of components \( \psi_{m'}(\vec{r}) \) satisfying Eq. (12).

By repeated insertion of Eq. (12) into Eq. (14) an expansion can be derived for \( f_n \). Obviously, the iteration will have practical significance only if the successive terms decrease sufficiently rapidly. Since this will not be true in general, it is important to establish the conditions under which the first nonvanishing
term of Eq.(14) represents a good approximation. For (a) \(\omega \gg |E_0(\alpha_0)|\), where \(E_0(\alpha_0)\) is the ground-state energy of the modified Hamiltonian Eq.(11) (Note that from Eq.(6) it follows that by increasing \(\alpha_0\), the potential \(V_0\) becomes shallower, and therefore \(|E_0(\alpha_0)| \) decreases from its unperturbed value at \(\alpha_0 = 0\): \(|E(\alpha)| < |E(0)|\)); (b) \(\alpha_0^2 \gg 1\); (c) \(\omega \gg E\), it was possible to extract the exact form of the dominant contribution to the last term of Eq.(14) (denoted below by \(f^{(1)}(\bar{r})\)) for an arbitrary potential of the type discussed before. In the case of elastic scattering we find

\[
\begin{pmatrix}
\text{Re} f_{10}(\bar{r}) \\
\text{Im} f_{10}(\bar{r})
\end{pmatrix} = \frac{z^2}{6\alpha_0 \omega^2} \left[ \frac{(-\bar{r}_0)\psi(\bar{r})\psi(\bar{r}_0) + \psi(\bar{r})\psi(\bar{r}_0)(-\bar{r}_0)}{\bar{p}_0\bar{p}_0} \right] \left( \frac{\alpha_0^2}{\pi \ln \alpha_0^2 \omega + \omega((\alpha_0^2 \omega)^0)} \right)
\]

(15)

where \(\bar{\bar{r}} = \bar{r}\) is the final momentum (see Eq.(3) for \(n = 0\)), and the corrective terms \(0\) also depend on \(\alpha_0\), \(E\), \(\omega\). Thus at fixed \(\alpha_0\) (this constraint appears also in the derivation of the Kroll and Watson result\(^2\)), and sufficiently high \(\omega\) (obeying conditions (a), (b), and (c) and the dipole-approximation assumption) it is possible to satisfy the inequality (d): \(|f_1^{(1)}(\alpha_0, \omega; E, \theta)| \ll |f_0^{(0)}(\alpha_0; E, \theta)|\). Whereas this holds in general over wide ranges of parameter values, it should nevertheless be checked for each case separately because \(f_1^{(0)}\) may become exceptionally small for certain angles. Conditions (a)-(d) together ensure the dominance of the first term in Eq. (14). However, this may hold under wider conditions than we were able to prove.

Thus, the elastic amplitude \(f_0\) reduces, to lowest order (in the sense discussed above), to \(f_0^{(0)}\), which is that calculated from the time-independent Schrödinger equation Eq.(13). This shows that in the high-frequency, high-intensity regime the incoming electron feels only the static distorted potential \(V_0(\alpha_0; \bar{r})\), the "dressed" potential associated to \(V(r)\):

\[
V_0(\bar{\alpha}_0, \bar{r}) = \frac{1}{\pi} \int_{-1}^{+1} V(\bar{r} + \bar{\alpha}_0 u) \frac{du}{\sqrt{1-u^2}}.
\]

(16)

The dressed potential, Eq.(16), can be looked upon as created by a linear distribution of "charges" extending from \(-\alpha_0\) to \(+\alpha_0\) along \(\hat{z}\), with density

\[
\sigma(z) = \frac{1}{\alpha_0} \left[ 1 - \left( \frac{z}{\alpha_0} \right)^2 \right]^{-\frac{1}{2}},
\]

(17)

the unit of "charge" generating the potential \(V(r)\). This behavior appears natural due to the rapid oscillations of the center of force in Eq.(1). In the regime we are considering, \(\omega\) and \(I\) enter the scattering problem only through \(\alpha_0\).

For the absorption amplitude \((n > 0)\) we get to lowest order from Eq.(14)
\[ f_n(\hat{\alpha}_0, \omega; \hat{p}) = -\frac{1}{2\pi} \left\langle \psi_\downarrow^{(-)} | V_n | \psi_\uparrow^{(+)} \right\rangle. \] (18)

Thus, the Fourier component \( V_n \) acts in our regime as a transition operator between scattering states \( \psi_\downarrow^{(\pm)} \) of the dressed potential \( V_0 \). The amplitude \( f_n \) depends on \( \omega \) via the final momentum \( \hat{p}_n \) (see Eq.(3)). Because \( \omega \) was assumed to be large at given \( \alpha_0 \), all \( f_n \) will be small with respect to \( f_0 \). (This contrasts with the low-frequency case where many \( f_n \) may be larger than \( f_0 \).) Note that the condition (c) above precludes free-free emission \( (n < 0) \).

III. ELASTIC SCATTERING FROM THE DRESSED COULOMB POTENTIAL

Since the original potential \( V(r) \) is spherically symmetric, \( V_0(\hat{\alpha}_0, \hat{r}) \) has axial symmetry around \( \hat{\alpha}_0 \) (we have assumed linear polarization). In the case of the pure Coulomb potential \( V(r) = -Z/r \), Eq.(16) yields

\[ V_0(\hat{\alpha}_0, \hat{r}) = -(2Z/\pi)(r_+r_-)^{-1/2} K(2^{-1/2}(1-r_+r_-)^{1/2}), \] (19)

where \( r_\pm = \hat{r} \mp \hat{\alpha}_0 \), and \( K \) is a complete elliptic integral of the first kind. \( V_0 \) has a logarithmic singularity along the distribution of charges, and \( r_\pm^{-1/2} \) singularities at its end points (all weaker than the original Coulomb singularity). The dressed Coulomb potential Eq.(19) is represented in Fig. 1.

The azimuthal symmetry of \( V_0 \) complicates the elastic scattering problem, as the azimuthal quantum number \( \ell \) is no longer conserved. Computations taking this into account have been performed by Van de Ree, Kaminski and Gavrila (to be published). In this section we shall describe a simplified approach in which \( V_0(\hat{\alpha}_0, \hat{r}) \) is modelled by its spherical average \( \bar{V}_0(\alpha_0, r) \). The problem is thus reduced to a phase shift calculation. This should give the right order of magnitude for the elastic scattering, particularly at low energies, when the electron wavelength is larger than the extension of the line of charges, i.e. \( \rho \alpha_0 \lesssim 1 \) (\( \rho \) is the electron momentum). On the other hand, interesting features related to the dependence of the cross section on the polarization vector \( \mathbf{\hat{e}} \) (or \( \mathbf{\hat{\alpha}_0} \)) will thus be lost.

By averaging Eq.(19) over all directions of \( \hat{\alpha}_0 \), one finds:

\[ \bar{V}_0(\alpha_0, r) = \frac{-Z}{\pi \alpha_0^3} \left\{ 2 \arcsin \rho - \rho \ln \frac{1 - (1 - \rho^2)^{1/2}}{1 + (1 - \rho^2)^{1/2}} \right\}, \quad \rho \leq 1 \]

\[ = -\frac{Z}{\alpha_0^3}, \quad \rho \geq 1, \] (20)

where \( \rho = r/\alpha_0 \). For \( r \geq \alpha_0 \), \( \bar{V}_0 \) coincides with the original Coulomb potential. For \( r < \alpha_0 \), \( \bar{V}_0 \) can be expanded as

\[ \bar{V}_0(\alpha_0, r) = -\frac{2Z}{\pi \alpha_0} \left( \ln \rho + \sum_{n=0}^{\infty} A_n \rho^{2n} \right), \] (21)
which displays a logarithmic singularity at the origin. This is much weaker than that of the Coulomb potential.

As is easily seen, $\tilde{V}_o(\alpha_o, r)$ is the potential energy due to a spherically symmetric distribution of electric charges extending up to $r = \alpha_o$, of radial density:

$$\tau(r) = 2Z \sigma(r),$$

(22)

with the function $\sigma$ defined by Eq.(17). Our scattering problem resembles thus that of an electron probing an extended charge nucleus of density $\tau(r)$ and radius $\alpha_o$; in contrast to this, however, we are dealing with a nonrelativistic case.

The determination of the continuum solutions of the potential $\tilde{V}_o$ of Eqs.(20), (21) raises some interesting mathematical points on account of its logarithmic
singularity at the origin. A number of results have been obtained lately for the bound state problem of the purely logarithmic case (all \( A_a = 0 \) in Eq.(21)), e.g. see Gesztesy and Pittner\(^{11} \), and Müller-Kirsten and Bose\(^{12} \). By expanding the \( \xi \)-th reduced radial partial wave as follows

\[
p_{\xi}(\rho) = \rho^{\xi+1} \sum_{m=0}^{\infty} (\rho^2 \ln \rho)^m \sum_{n=0}^{\infty} a_{mn}^{(\xi)} \rho^{2n},
\]

we have uniquely determined the coefficients \( a_{mn}^{(\xi)} \) by recurrence relations. We have checked numerically the convergence of the series for small \( \rho \). On account of its behaviour for \( \rho \to 0 \), it represents the physically acceptable solution. It can be used to start the numerical solution in the vicinity of the origin. Beyond \( \rho = 1 \) the solution goes over into a linear combination of the regular and irregular Coulomb functions, and the extra phase shifts \( \delta_{\xi} \), due to the deviation of \( V_0 \) from a Coulomb potential, can be determined.

For an attractive potential \( (Z > 0) \) \( \delta_{\xi} \) is negative, and for given \( \xi \) and \( E \) at small values \( \alpha_o \), its dominant behaviour reads:

\[
\delta_{\xi} = \gamma \frac{2^{2\xi} e^{-\pi \gamma/4} \Gamma(\xi+1+1/2)^2}{[2(\xi+1)]^{1/2}} \frac{\Gamma(\xi+3/2)(p\alpha_o)^{2\xi+2}}{(\xi+1)(\xi+3)^{1/2} \sqrt{\pi}} \cdot \left( 1+O(\alpha_o) \right),
\]

in which \( \gamma = -Z/p \). For values \( \alpha_o \leq 10^{-2} \), Eq.(24) agrees quite well with the numerical computation.

For large values of \( \alpha_o \), on the other hand, we find analytically (at given \( \xi \) and \( E \)) the behaviour:

\[
\delta_{\xi} = \gamma \ln \alpha_o + x_{\xi}(p) + O(1/\alpha_o),
\]

where \( x_{\xi}(p) \) is independent of \( \alpha_o \). Eq.(25) can be derived from an extension of the usual JWKB phase shift expression to the case of a modified Coulomb potential like that of Eq.(20). It is limited by the condition: \( p\alpha_o \gg \xi + 1/2 \). Note that, although \( p \) is fixed and can be small in our case, the JWKB approximation still applies, because at large \( \alpha_o \) the magnitude of the potential \( V_0 \) (and its derivatives) becomes small, while its range increases as \( \alpha_o \). The logarithmic build-up of \( \delta_{\xi} \) with \( \alpha_o \) displayed by Eq.(25), is peculiar to the Coulomb long-range behaviour of \( V_0 \). For \( \alpha_o \geq 10^{-2} \), this was quite well checked numerically.

Because of the long range behaviour of the potential, Eq.(20), the scattering amplitude is given by the theory for modified Coulomb scattering:

\[
f(\theta) = f_c(\theta) + f'(\theta),
\]

\[
f_c(\theta) = \frac{(-\gamma)}{2p \sin^2 \frac{\theta}{2}} \exp(-i\gamma \ln \sin^2 \frac{\theta}{2} + 2i\alpha_o).
\]
\[ f'(\theta) = \frac{1}{2i\rho} \sum_{k} (2i\lambda+1)e^{2i\phi_k} (e^{2i\delta_k} - 1) P_k(\cos \theta). \] (28)

Here \( \theta \) is the scattering angle, \( f_C \) is the Coulomb amplitude, \( f' \) is the extra contribution due to \( \tilde{V}_o - V \), and \( \phi_k \) are the Coulomb phases. The modified cross section is

\[ \frac{d\sigma}{d\Omega} = \frac{d\sigma_C}{d\Omega} + 2 \text{Re} f_C^* f' + |f'(\theta)|^2, \] (29)

where the first term on the right is the Rutherford cross section and the second represents the interference of the Coulomb and short-range amplitudes. Eqs. (26)-(29) are similar to those used in the analysis of the classical proton-proton collision experiments at low (nuclear) energies with the short range nuclear force taken into account; see Ref. 13, chapter 10, §§ 5 and 9 (exchange effects are absent here).

For small \( \alpha_o \), Eq. (24) shows that only the \( \lambda = 0 \) phase should be considered; to lowest order, \( \delta_0 = 0 (\alpha_o^2) \). The cross section, Eq. (29), can then be cast into the form used for proton-proton scattering (with exchange neglected), see Ref. 13, chapter 10, equation (5.1).

In Fig. 2 we present the angular dependence of the ratio \( R = (d\sigma/d\Omega)/(d\sigma_C/d\Omega) \) of the modified cross section, Eq. (29), to the Rutherford cross section at \( Z = 1 \) for a number of electron energies \( E \) and values of \( \alpha_o \). The energies \( E \) chosen satisfy the condition of validity (b) of our theory given in Sec. II, for \( \omega \) attainable with some existing high-frequency lasers (see Luk et al.\(^5\), Rhodes\(^6\)). The values considered for \( \alpha_o \) have also been achieved experimentally. (When the laser of Luk et al.\(^5\) is operated at \( I = 10^{15} \) W/cm\(^2\), we have \( \alpha_o = 3.1 \) and at \( I = 10^{16} \) W/cm\(^2\), \( \alpha_o = 9.8 \)).

In all cases \( R \rightarrow 1 \) as \( \theta \rightarrow 0. \) This is due to the fact that \( f_C(\theta) \) and \( d\sigma_C/d\Omega \) become infinite as \( \theta \rightarrow 0 \), whereas \( f'(\theta) \) stays finite. Moreover, for \( \theta \rightarrow 0 \), \( R \) has infinitely many oscillations of increasing frequency and decreasing amplitude. These are due to the presence of the Coulomb phase factor, see Eq. (27), in the interference term of Eq. (29) (but not in \( d\sigma_C/d\Omega \)). Indeed, by setting \( \gamma = 0 \) in the phase factor, the oscillations disappear and \( R \) either becomes a monotonically decreasing function of \( \theta \) (for small \( \alpha_o \)), or has a broad maximum and then decreases (for larger \( \alpha_o \)).

The amplitude of these Coulomb-interference oscillations decreases for \( \theta \rightarrow 0 \) because the relative importance of the interference term in Eq. (29) diminishes compared to \( d\sigma_C/d\Omega \). Note that, as seen in Fig. 2, the Coulomb-interference oscillations occur in an accessible range of experimental parameters. (They were not detected in the proton-proton collision experiments, since there the energy had to be high enough, \( \gamma \approx 0 \), to overcome the Coulomb repulsion so that the particles could approach within the range of nuclear forces).
Fig. 2. Ratio $R$ of the elastic scattering cross section for the averaged dressed Coulomb potential, Eq. (20), to the Rutherford cross section. Nuclear charge $Z = 1$ Values of the electron energy $E$ (in Ry) and of $\alpha_o$, as indicated (for the definition of $\alpha_o$ see Eq. (2)).
For large scattering angles Fig. 2 shows that the modified cross section is considerably reduced with respect to the original value, because of destructive interference in Eq.(29).

In Fig. 3 we display the effect on $R$ of the increase of the nuclear charge $Z$, at given $E$ and $\alpha_0$. It is apparent that the Coulomb oscillations will extend to higher angles, and become quite regular. For $Z = -1$ (positron-proton scattering) and the parameters of Fig. 3, $R = 1$ for all angles (not drawn). This happens because in this case the positron wave function does not penetrate into the region ($r \leq \alpha_0$) where the potential is modified by the laser field, and the scattering remains purely Coulombic.

*Fig. 3.* Ratio of cross sections $R$ for $Z = 1$ and $Z = 10$, at $E = 0.05$ Ry and $\alpha_0 = 5$. 